Investigation of Numerical Stability of 2D FE/FDTD Hybrid Algorithm for Different Hybridization Schemes

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SUMMARY  Numerical Stability of the Finite Element/Finite Difference Time Domain Hybrid algorithm is dependent on the hybridization mechanism adopted. A framework is developed to analyze the numerical stability of the hybrid time marching algorithm. First, the global iteration matrix representing the hybrid algorithm following different hybridization schemes is constructed. An analysis of the eigenvalues of this iteration matrix reveals the stability performance of the algorithm. Thus conclusions on the performance with respect to numerical stability of the different schemes can be arrived at. Further, numerical experiments are carried out to verify the conclusions based on the stability analysis.

key words: FE/FDTD, numerical stability, hybrid algorithms

1. Introduction

The Finite-Difference Time-Domain (FDTD) Method is one of the efficient time domain methods widely used in studying transient and wide-band phenomena in various electromagnetic and microwave systems. Stair-case approximation of the boundary of the geometry modeled is a major source of errors in the FDTD algorithm. To overcome stair-casing errors, the FDTD algorithm has been hybridized with the finite element method (FEM) in the time domain [1], [2]. This hybrid algorithm facilitates the accurate modeling of geometries by meshing the region in the vicinity of the geometry using unstructured grids conforming to the geometry, while other parts of the physical domain is modeled using traditional FDTD method with Cartesian grids. In this communication, we compare the numerical stability of the Finite-Element/Finite-Difference Time-Domain (FE/FDTD) hybrid algorithm for four different hybridization schemes. The construction of the global iteration matrix representing the hybrid time-marching algorithm will be shown. An eigenvalue analysis of this global iteration matrix is performed to compare and analyze the numerical stability of the different hybridization schemes. Numerical experiments are performed and good agreement with the conclusions based on the stability analysis are observed.

2. Hybrid Algorithm Formulation

The original FE-FDTD method proposed in [2] splits the physical space into two overlapping domains viz., the Finite Difference, $\Omega_{FD}$ and Finite Element, $\Omega_{FE}$ regions. The overlapping region is one FDTD cell thick. Traditional leapfrog scheme on a staggered grid is used for the unknowns in $\Omega_{FD}$. Edge vector basis functions over triangular elements with the unconditionally stable Newmark-\(\beta\) scheme is used for the temporal discretization of the unknowns in $\Omega_{FE}$. Scheme I is that used in [2], [4] and an example of the hybrid mesh for this scheme is shown in Fig. 1. Scheme II is similar to that in [2], where the overlapping region with thickness of one FDTD cell, between the two domains is triangulated into four triangular elements as shown in Fig. 2, instead of two as in Scheme I. Scheme III follows the strategy proposed in [5] and the hybrid mesh for this case is shown in Fig. 3. In [6], Scheme II was shown to introduce less un-
physical reflections in the numerical solution and hence has better accuracy compared to Scheme I and III. In Scheme IV, the overlapping region is modeled using rectangular edge elements and the rest of $\Omega_{FE}$ is modeled using triangular finite elements. Example mesh for this scheme is shown in Fig. 4 where the shaded cells are the rectangular elements in $\Omega_{FE}$. It is easier to generate the hybrid mesh for Scheme IV as compared to Schemes I and II.

2.1 Stability Analysis

Temporal instabilities often arise in the FE-FDTD hybrid algorithm [2], [4]. In [7] filtering techniques are proposed to control the stability of the algorithm. It is very interesting to note that the Newmark-$\beta$ scheme, which is equivalent to the $\theta$-method discussed in [3], is unconditionally stable for $\beta \geq 0.25$. The FDTD algorithm by itself is conditionally stable. However, the hybrid algorithm is often unstable. The following framework sheds some insight on the stability behaviour of the hybrid algorithm with various possible hybridization schemes. The analysis to follow is for the 2D $TE_z$ case. However, the application of the same to the 2D $TM_z$ case is straightforward.

The numerical stability of a time-marching algorithm represented by

$$v^{n+1} = G(\Delta t, \Delta h)v^n$$  

(1)

where $v^n$ is the unknown at time $n\Delta t$ can be investigated by analyzing the eigenvalues of the global iteration or amplification matrix $G$. The necessary condition for stability is $\rho(G) \leq 1$, where $\rho(G)$ is the spectral radius or magnitude of the largest eigenvalue of the matrix $G$ [8]. To obtain $G$ for the hybrid algorithm, the update equations for the unknowns in $\Omega_{FD}$ and $\Omega_{FE}$ have to be combined. It is noted that due to the unstructured nature of the finite element mesh, it is difficult to compute the eigenvalues of $G$ analytically. The notation used in the following analysis is shown in Table 1.

Consider the magnetic and electric field FDTD update equations for the unknowns in $\Omega_{FD}$, as follows

$$b_{FD}^{n+1} = b_{FD}^{n-1/2} + A_{FD}^t b_{FD}^n$$

(2)

$$\begin{bmatrix} e_{FD}^{n+1} \\ e_{FE}^{n+1} \\ e_{FD}^{n-1/2} \end{bmatrix} = \begin{bmatrix} A_{FD}^t & e_{FD}^n \\ e_{FD}^n & e_{FD}^{n-1/2} \end{bmatrix}$$

(3)

In general the matrix $A_{FD}^t$ and $A_{FD}^t$ are sparse. For Maxwell’s equations in the lossless case, the non-zero entries of $A_{FD}^t$ is $\pm \Delta t/(\mu_0 \Delta h)$ and that of $A_{FD}^t$ is $\pm \Delta t/(\epsilon_0 \Delta h)$. Here, $\Delta t$ is the time step size and $\Delta h$ is the space step size. $A_{FD} = A_{FD}^t A_{FD}^t$ can be viewed as the discrete ($\Delta t^2/c^2) \nabla \times \nabla \times$ operator. By eliminating $h_{FD}$ from the FDTD update equations, the electric field update equation in $\Omega_{FD}$ is

\[
\begin{bmatrix} e_{FD}^{n+1} \\ e_{FE}^{n+1} \\ e_{FD}^{n-1/2} \end{bmatrix} = \begin{bmatrix} 2I + A_{11} & A_{12} & 0 \\ 0 & 2I + A_{11} & A_{23} \\ 2I + A_{11} & 2I + A_{11} & 2I + A_{11} \end{bmatrix} \begin{bmatrix} e_{FD}^n \\ e_{FE}^n \\ e_{FD}^{n-1/2} \end{bmatrix} - \begin{bmatrix} e_{FD}^{n-1/2} \\ e_{FE}^{n-1/2} \\ e_{FD}^{n-1/2} \end{bmatrix}
\]

(4)

where

$$A_{FD} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{11} & A_{23} \\ 0 & A_{32} & A_{11} \end{bmatrix} = A_{FD}^t A_{FD}^t$$

(5)

The implicit update equations for the unknowns in $\Omega_{FE}$ is written as

$$\begin{bmatrix} M_{33} & M_{32} & M_{31} \\ M_{32} & M_{32} & M_{32} \\ M_{31} & M_{32} & M_{31} \end{bmatrix} \begin{bmatrix} e_{FD}^{n+1} \\ e_{FE}^{n+1} \\ e_{FD}^{n-1/2} \end{bmatrix} = \begin{bmatrix} N_{33} & N_{32} & N_{31} \\ N_{32} & N_{32} & N_{32} \\ N_{31} & N_{31} & N_{31} \end{bmatrix} \begin{bmatrix} e_{FE}^n \\ e_{FE}^n \\ e_{FE}^{n-1/2} \end{bmatrix}$$

(6)

Table 1: Notations used for stability analysis.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b_{FD}$</td>
<td>Magnetic field unknowns in $\Omega_{FD}$</td>
</tr>
<tr>
<td>$e_{FD}$</td>
<td>Electric field unknowns in $\Omega_{FD}$</td>
</tr>
<tr>
<td>$e_{FE}$</td>
<td>Electric field unknowns in $\Omega_{FE}$</td>
</tr>
<tr>
<td>$e_{FD}^{n+1}$</td>
<td>Electric field unknowns on the boundary of $\Omega_{FD}$</td>
</tr>
<tr>
<td>$e_{FE}^{n+1}$</td>
<td>Electric field unknowns on the boundary of $\Omega_{FE}$</td>
</tr>
</tbody>
</table>
The matrices \( M \) and \( N \) are given as

\[
M = \begin{pmatrix}
M_{33} & M_{23} & M_{13} \\
M_{23} & M_{22} & M_{12} \\
M_{13} & M_{12} & M_{11}
\end{pmatrix}, \quad N = \begin{pmatrix}
N_{33} & N_{23} & N_{13} \\
N_{23} & N_{22} & N_{12} \\
N_{13} & N_{12} & N_{11}
\end{pmatrix}
\]

where \( T \) and \( S \) are traditional finite element mass and stiffness matrices respectively. \( \beta \) is the parameter in Newmark-\( \beta \) scheme, which is unconditionally stable for \( \beta \geq 0.25 \).

The update equation for the hybrid algorithm can then be written by combining Eqs. (4) and (6) as,

\[
Q_1 e^{n+1} = Q_0 e^n - Q_1 e^{n-1}
\]

where

\[
e = \begin{pmatrix}
e_{FD} \\
e_{F} \\
e_{E}
\end{pmatrix}, \quad Q_1 = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & M_{23} & M_{22} & M_{12}
\end{pmatrix}, \quad Q_0 = \begin{pmatrix}
2I + A_{11} & A_{12} & 0 & 0 \\
A_{12}^t & 2I + A_{22} & A_{23} & 0 \\
0 & N_{23} & N_{22} & N_{21}
\end{pmatrix}
\]

Equation (7) can be written in the form of Eq. (1) using the following substitution,

\[
v^0 = \begin{pmatrix}
e^0 \\
e_{n-1}
\end{pmatrix},
\]

which results in the following iteration matrix,

\[
G = \begin{pmatrix}
Q_1^{-1} Q_0 & -I \\
I & 0
\end{pmatrix}
\]

The eigenvalues of \( G \), \( \lambda_G \), can be computed efficiently by computing the eigenvalues of a lower order matrix \( Q_1^{-1} Q_0 \) say \( \lambda_Q \). It can be shown that \( \lambda_G \) can be computed from \( \lambda_Q \) as

\[
\lambda_G = \frac{\lambda_Q}{2} \pm \sqrt{\left(\frac{\lambda_Q}{2}\right)^2 - 1}
\]

Upon assembling the matrices \( Q_0 \) and \( Q_1 \) for a given hybrid mesh, the eigen values of the iteration matrix \( \lambda_G \) and hence \( \rho(G) \) can be computed. \( \rho(G) \leq 1 \) is the necessary condition for stability (though not sufficient, since \( G \) is not a normal matrix [8]).

3. Numerical Experiments

The assembly of \( Q_0 \) and \( Q_1 \) is straightforward in the case of Schemes I, II and IV, since the unknowns \( e_{FD} \) and \( e_{FE} \) conform to each other. However, this is not the case for Scheme III, which is known to lead to instabilities [5]. Eigenvalue analysis is not performed on this scheme. It is worth noting that for Scheme II, \( M_{33} = N_{33} = 0 \) i.e., there is no coupling between \( e_{FD} \) and \( e_{FE} \). This is because of the discretization scheme used in the overlapping regions. Similarly in the case of Scheme IV, \( M_{13} = N_{13} = 0 \). The iteration matrix are constructed and its eigenvalues are computed for the case of hybrid meshes shown in Figs. 1, 2 and 4. The distribution of eigenvalues are shown in Figs. 5, 6 and 7. Ideally, all eigenvalues should lie on the unit circle in the complex plane. If any of the eigenvalue is inside the unit circle, then the numerical scheme is dissipative (which should not be the case when dealing with Maxwell’s equations in lossless media).

If any of the eigenvalue is outside the unit circle, then the scheme is numerically unstable. In Table 2, we compare the spectral radius and the percentage of eigenvalues lying outside the unit circle for the three schemes. It is clearly seen...
Table 2: Eigenvalue statistics of the iteration matrix in different schemes.

<table>
<thead>
<tr>
<th>Scheme I</th>
<th>Scheme II</th>
<th>Scheme IV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\rho(G)$</td>
<td>1.04423</td>
<td>1.030587</td>
</tr>
<tr>
<td>% of $</td>
<td>\lambda_G</td>
<td>&gt; 1$</td>
</tr>
</tbody>
</table>

that the spectral radius of the iteration matrix for Scheme IV is closer to unity and hence has better stability performance over the other schemes.

To verify the conclusions on stability based on the above analysis, we compute the $TE_z$ scattering by a perfectly conducting (PEC) cylinder using the four hybridization schemes. A differentiated Gaussian pulse with spectral content in the band 0.8–2 GHz is incident along the $\hat{x}$-direction. The backscattered $H_z(t)$ component is shown in Fig. 8. It is clearly observed that instabilities arise within 1,000 time steps for Schemes I, II and III, while for Scheme IV, it starts to appear around 60,000 time steps. Thus, Scheme IV has better numerical stability, as concluded in the eigenvalue analysis.

The second example is the 2D $TE_z$ scattering from a NACA64a410 airfoil. The length of the airfoil is 1 m which corresponds to $5\lambda$ at 1.5 GHz. The hybrid mesh used in this computation is shown in Fig. 9 where the fine triangular discretization along the trailing edge of the airfoil can be observed. For triangulation in $\Omega_{FE}$, the Triangle code [10] has been used. The incident plane wave is a $\hat{x}$-directed differentiated Gaussian pulse with significant spectral content in the band of 0.2–1.5 GHz. The FDTD grid size is set as $\Delta x = \Delta y = 0.01$ m. A Total-Field/Scattered-Field boundary is implemented in $\Omega_{FD}$. A time domain near-to-far-field transformation (NFFT) [9] is performed on the near fields in the scattered field region to obtain far zone scattered electric field components. For the implicit update equation on unknowns in $\Omega_{FE}$, a preconditioned conjugate gradient solver is used with a complete Cholesky factor as a preconditioner. Note that the preconditioner needs to be contracted only once for all time steps, before the time marching begins.

The 2D backscattered RCS over the frequency range of 0.2–1.5 GHz is shown in Fig. 10. The results are compared with the frequency domain based method of moment solution [11]. Good agreement is observed across the entire band. In Fig. 11 the bistatic RCS pattern at 1.5 GHz, obtained using the hybrid algorithm with Scheme IV, is in agreement with the method of moment solution. Deviations in the solution could be attributed to the relatively coarse cell size of $\lambda/20$ at 1.5 GHz used in $\Omega_{FD}$ as compared to the $\lambda/80$ mesh size used in the MoM computation.

Since the number of time steps reflects the resolution in the frequency domain, Scheme IV should be used to analyze problems with high quality factor (which demand high frequency resolution).
frequency resolution, and hence more time steps in the time domain solution).

4. Conclusion

A framework based on the eigenvalue analysis to study the numerical stability of hybrid FE/FDTD algorithm is developed. Construction of global iterative matrix for the hybrid algorithm is presented. The eigenvalue analysis of the iteration matrix for different hybridization schemes revealed that Scheme IV has better stability properties than others. This was further validated by numerical experiments using various hybridization strategies.

References


